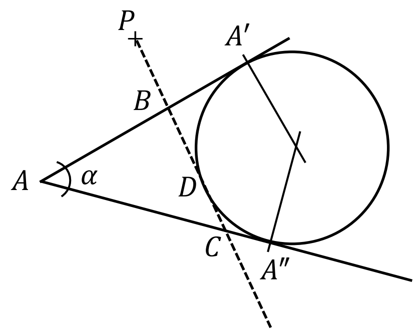
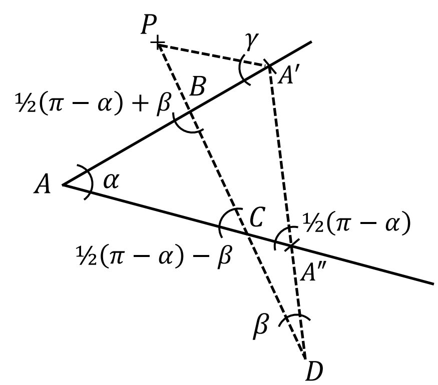
Problem 19) Mark the point A'' on the second leg of the angle such that $\overline{AA'} = \overline{AA''}$. Draw a circle inside the angle that is tangent to both its legs at A' and A''. From the point P, draw a tangent to the circle. This tangent, which meets the circle at point D, is the desired straight line. Note that $\overline{BD} = \overline{BA'}$ and $\overline{CD} = \overline{CA''}$. Consequently,

$$\text{Perimeter of } ABC = (\overline{AB} + \overline{BD}) + (\overline{AC} + \overline{CD}) = \overline{AA'} + \overline{AA''} = 2\overline{AA'}.$$



Digression: The analytic solution of the problem is far more complicated. To see this, choose the point A'' on the lower leg of the angle α such that $\overline{AA''} = \overline{AA'}$. If the straight line from P goes through A'', the perimeter of the ABC triangle will exceed $\overline{AA'} + \overline{AA''}$, because $\overline{BC} = \overline{BA''}$ will be greater than $\overline{BA'}$. Therefore, the line through P must cross $A'A''$ somewhere below the lower leg, say, at D. The perimeter of the ABC triangle will now equal twice $\overline{AA'}$ if $\overline{BC} = \overline{BA'} + \overline{CA''}$. Invoking the triangle property that the ratio of each side to the sine of its opposite angle is the same for all sides, and with reference to the figure, we now write



$A'BD$ triangle: $\quad \overline{BD}/\sin[\tfrac{1}{2}(\pi-\alpha)] = \overline{BA'}/\sin\beta. \qquad (1)$

$A''CD$ triangle: $\quad \overline{CD}/\sin[\tfrac{1}{2}(\pi-\alpha)] = \overline{CA''}/\sin\beta. \qquad (2)$

Subtracting the second of the above equations from the first, we find

$$\overline{BA'} + \overline{CA''} = \overline{BC} = \overline{BD} - \overline{CD} = \frac{\cos(\alpha/2)}{\sin\beta}(\overline{BA'} - \overline{CA''}) \quad \rightarrow \quad \overline{BA'}/\overline{CA''} = \frac{\cos(\alpha/2) + \sin\beta}{\cos(\alpha/2) - \sin\beta}. \qquad (3)$$

Next, we write down the relevant identities for the ABC triangle and, with the aid of Eq.(3), obtain expressions for $\overline{BA'}$ and $\overline{CA''}$, as follows:

ABC triangle: $\quad \overline{AB}/\sin[\tfrac{1}{2}(\pi-\alpha)-\beta] = \overline{AC}/\sin[\tfrac{1}{2}(\pi-\alpha)+\beta] = \overline{BC}/\sin\alpha$

$$\rightarrow \quad \overline{AA'} - \overline{BA'} = \frac{\cos(\tfrac{1}{2}\alpha+\beta)}{\sin\alpha}(\overline{BA'} + \overline{CA''}), \qquad (4a)$$

$$\text{and} \quad \overline{AA''} - \overline{CA''} = \frac{\cos(\tfrac{1}{2}\alpha-\beta)}{\sin\alpha}(\overline{BA'} + \overline{CA''}). \qquad (4b)$$

Adding Eqs.(4a) and (4b) now yields

$$2\overline{AA'} - (\overline{BA'} + \overline{CA''}) = \frac{2\cos(\alpha/2)\cos\beta}{\sin\alpha}(\overline{BA'} + \overline{CA''}) = \frac{\cos\beta}{\sin(\alpha/2)}(\overline{BA'} + \overline{CA''})$$

$$\rightarrow \quad \overline{BA'} + \overline{CA''} = \left[\frac{2\sin(\alpha/2)}{\sin(\alpha/2) + \cos\beta}\right]\overline{AA'}. \qquad (5)$$

Upon substitution from Eq.(3) into Eq.(5), we find

$$\overline{BA'} = \left\{\frac{[\cos(\alpha/2)+\sin\beta]\tan(\alpha/2)}{\sin(\alpha/2) + \cos\beta}\right\}\overline{AA'}, \qquad (6a)$$

$$\overline{CA''} = \left\{\frac{[\cos(\alpha/2) - \sin\beta]\tan(\alpha/2)}{\sin(\alpha/2) + \cos\beta}\right\}\overline{AA'}. \qquad (6b)$$

Finally, from the triangles PDA' and BDA', we obtain

$$\overline{PA'}/\sin\beta = \overline{DA'}/\cos(\gamma + \beta - ½\alpha), \qquad (7a)$$

$$\overline{BA'}/\sin\beta = \overline{DA'}/\cos(\beta - ½\alpha). \qquad (7b)$$

Consequently,

$$\overline{PA'}/\overline{BA'} = \frac{\cos(\beta - ½\alpha)}{\cos(\gamma + \beta - ½\alpha)} \quad \rightarrow \quad \overline{PA'}/\overline{AA'} = \frac{\cos(\beta - ½\alpha)[\cos(\alpha/2) + \sin\beta]\tan(\alpha/2)}{\cos(\gamma + \beta - ½\alpha)[\sin(\alpha/2) + \cos\beta]}. \qquad (8)$$

Given that the lengths $\overline{AA'}$ and $\overline{PA'}$ as well as the angles α and γ are known, Eq.(8) may be solved for the desired angle β.
